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The Geometry of Constrained Motion

Die Arbeit bringt eine geometrische Interpretation der expliziten Bewegungsgleichungen für mechanische Systeme mit Zwangsbedingungen. Dies führt zu einem geometrischen Prinzip der analytischen Mechanik.

This paper provides a geometrical interpretation of the explicit equations of motion for constrained mechanical systems. This leads to a geometric principle of analytical mechanics.

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The geometry of constrained motion

The principles of analytical mechanics given by D'ALEMBERT, LAGRANGE, and GAUSS are all-encompassing and therefore it is natural that there cannot be a new fundamental principle for the theory of motion and equilibrium of discrete, dynamical systems. However, additional perspectives are often useful if they can help provide deeper insights and/or help solve specific problems of special importance. It is in this vein that in this paper we provide a basic geometric interpretation of constrained motion which results in the statement of a geometric principle underlying analytical dynamics.

Consider a set of n point-particles of masses $m_1, m_2, \ldots, m_n$ whose positions are described in a Cartesian coordinate frame of reference by the $3n$-vector $\boldsymbol{x} = [x_1, x_2, \ldots, x_{3n}]^{\mathrm{T}}$. If the components of the external forces acting on the particles are given by the $3n$-vector $\boldsymbol{F} = [F_1, F_2, \ldots, F_{3n}]^{\mathrm{T}}$, then the unconstrained equation of motion of the system can be written as

$$M\ddot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t), \tag{1}$$

where the matrix M is a known $3n$ by $3n$ diagonal, positive definite matrix containing the masses of the n particles in sets of threes along the diagonal. Specifically, by "unconstrained" we mean that the number of degrees of freedom of the system equals the minimum number of coordinates needed to define the system's configuration. The acceleration of the unconstrained system is therefore given, using equation (1), by

$$\boldsymbol{a} = M^{-1}\boldsymbol{F}. \tag{2}$$

Further, let this system be constrained by the m consistent, though not necessarily independent, constraints

$$\varphi_i(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = 0, \qquad i = 1, 2, \ldots, m. \tag{3}$$

These constraints include both holonomic and nonholonomic contraints. Assuming that the φ_i's are sufficiently smooth, we can appropriately differentiate the equation set (3) with respect to time t, to give

$$A(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)\,\ddot{\boldsymbol{x}} = \boldsymbol{b}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t), \tag{4}$$

where A is an m by $3n$ matrix. Each holonomic constraint of the set (3) would need to be differentiated twice with respect to time while each nonholonomic constraint would need only one differentiation to put equation set (3) in the form of equation (4).

Given a set of initial conditions $\boldsymbol{x}(t_0)$ and $\dot{\boldsymbol{x}}(t_0)$ which satisfy the constraint equations (3), the explicit equation of motion describing the time-evolution of the constrained system which we have described above can be written as (UDWADIA and KALABA [1])

$$\ddot{\boldsymbol{x}} = \boldsymbol{a} + M^{-1/2}C^{+}\boldsymbol{e}, \tag{5}$$

where, the constraint matrix C is defined to be $AM^{-1/2}$, the vector $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{a}$, and the superscript "+" denotes the Moore-Penrose inverse of the matrix C.

If the singular value decomposition of the matrix C is expressed as $U\Lambda V^{\mathrm{T}}$, then the Moore-Penrose inverse of C is given by $C^{+} = V\Lambda^{-1}U^{\mathrm{T}}$. This (unique) matrix C^{+} is such that $C^{+}CC^{+} = C^{+}$, $CC^{+}C = C$, $(CC^{+})^{\mathrm{T}} = CC^{+}$, and $(C^{+}C)^{\mathrm{T}} = C^{+}C$.

In order to geometrically interpret the motion of the constrained system, we begin by premultiplying equation (5) by $M^{1/2}$. Denoting the "scaled accelerations" $M^{1/2}\ddot{\boldsymbol{x}}$ and $M^{1/2}\boldsymbol{a}$ by $\ddot{\boldsymbol{x}}_s$ and $\boldsymbol{a}_s$, respectively, we rewrite equation (5) as

$$\ddot{\boldsymbol{x}}_s = \boldsymbol{a}_s + C^{+}\boldsymbol{e}. \tag{6}$$

Here $\ddot{\boldsymbol{x}}_s$ and $\boldsymbol{a}_s$ correspond to the scaled accelerations of the unconstrained and the constrained system, respectively, at time t.

We begin by stating our **main result**:

The motion of the constrained system of particles described above proceeds at each instant of time in such a manner that

(1) *the orthogonal projection of the scaled acceleration of the constrained system on the column space of C^{T} exceeds that of the scaled acceleration of the unconstrained system by C^+e, where C is the constraint matrix and e is the extent to which the unconstrained acceleration a does not satisfy, at that instant of time, the constraint equation* (4); *and,*

(2) *the orthogonal projections of the scaled accelerations of the constrained and the unconstrained system on the null space of C are identical.*

We now present the following results which we will need in order to prove this geometrical feature of constrained motion.

Result 1: *The matrix $P = C^+C$ is an orthogonal projection on the space spanned by the columns of C^{T}. We will denote this space as $\mathscr{R}(C^{\mathrm{T}})$.*

Proof: Let the matrix A have rank r. Then the singular value decomposition of the m by $3n$ matrix C whose rank is also r, is given by $C = U\Lambda V^{\mathrm{T}}$, where the r by r diagonal matrix Λ contains the singular values of C. Hence $P = C^+C = V\Lambda^{-1}U^{\mathrm{T}}U\Lambda V^{\mathrm{T}} = VV^{\mathrm{T}}$, so that

$$P^{\mathrm{T}} = P\,, \quad \text{and} \quad P^2 = P\,. \tag{7}$$

We next show that the space spanned by the columns of P is identical to that spanned by the columns of C^{T}. Thus we need to show that for any m-vector λ, there exists a $3n$-vector y such that

$$Py = C^{\mathrm{T}}\lambda\,. \tag{8}$$

A necessary and sufficient condition for the equation set (8) to be consistent is that

$$PP^+(C^{\mathrm{T}}\lambda) = C^{\mathrm{T}}\lambda\,. \tag{9}$$

Considering the left hand member of the above equation, we get

$$\begin{aligned} PP^+C^{\mathrm{T}} &= C^+C(C^+C)^+\,C^{\mathrm{T}} = C^+CC^+CC^{\mathrm{T}} = C^+CC^{\mathrm{T}} \\ &= (C^+C)^{\mathrm{T}}\,C^{\mathrm{T}} = C^{\mathrm{T}}(C^+)^{\mathrm{T}}\,C^{\mathrm{T}} = (CC^+C)^{\mathrm{T}} = C^{\mathrm{T}}\,, \end{aligned} \tag{10}$$

and hence (9) is proved. In view of (7) and (8), the result now follows.

Result 2: *The matrix $Q = [I - P] = [I - C^+C]$ is an orthogonal projection onto the null space of the matrix C. We denote the null space of C by $\mathscr{N}(C)$.*

Proof: Since P is symmetric, Q is symmetric. Also, $Q^2 = I - 2P + P^2$, which by (7) indicates that $Q^2 = Q$. We next need to show that for any $3n$-vector y, $C(Qy) = \mathbf{0}$. But we have,

$$CQ = U\Lambda V^{\mathrm{T}}(I - VV^{\mathrm{T}}) = (C - C) = \mathbf{0}\,, \tag{11}$$

and hence Q is an orthogonal projection onto the null space of C.

Result 3: *Any $3n$-vector y can be expressed as the sum of its orthogonal projections on $\mathscr{R}(C^{\mathrm{T}})$* and *$\mathscr{N}(C)$.*

Proof: The subspaces $\mathscr{R}(C^{\mathrm{T}})$ and $\mathscr{N}(C)$ are orthogonal complements of each other in the $3n$-dimensional vector space $\mathbb{R}^{3n}$. If the rank of the matrix A is r, then the dimensions of the subspaces $\mathscr{R}(C^{\mathrm{T}})$ and $\mathscr{N}(C)$ are r and $3n - r$, respectively.

Now premultiplying equation (6) by $P = C^+C$ we get

$$P\ddot{x}_s - Pa_s = C^+CC^+e = C^+e\,. \tag{12}$$

Noting Result 1, equation (12) proves the first part of our main result. We note that the right hand side of equation (12) can also be expressed as

$$C^+e = C^+(b - Aa) = C^+(b - Ca_s) = C^+b - Pa_s\,. \tag{13}$$

Thus equation (12) simplifies to

$$P\ddot{x}_s = C^+b\,. \tag{14}$$

The first part of our main result can then be restated as follows: *The projection of the acceleration of the constrained system of particles onto the subspace $\mathscr{R}(C^{\mathrm{T}})$ is C^+b.*

Equation (6) can also be written using (4) and our definitions of $\ddot{x}_s$ and a_s as

$$\ddot{x}_s - a_s = C^+(C\ddot{x}_s - Ca_s)\,, \tag{15}$$

from which it follows that

$$Q\ddot{x}_s = Qa_s, \tag{16}$$

where $Q = I - P$. Noting Result 2, this proves the second part of our main result.

Our geometrical interpretation of constrained motion lends itself to the following pictorial representation.

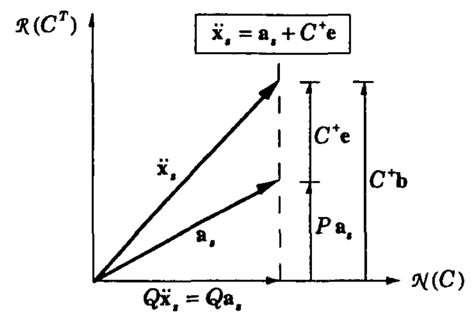

Fig. 1. The geometry of constrained motion. $P = C^+C$ and $Q = I - P$

To understand this pictorial description better, we can write the equation of motion of the constrained system as

$$\ddot{x}_s = a_s + M^{-1/2}F^c, \tag{17}$$

where the $3n$-vector F^c is the force of constraint which is brought into play by the presence of the constraint which is described by equation (4). Noting our definition of $\ddot{x}_s$, equation (4) can in turn be expressed as

$$C\ddot{x}_s = b. \tag{18}$$

Let us denote the scaled force of constraint, $M^{-1/2}F^c$, by F_s^c. The principle of virtual work then requires that for any scaled virtual displacement vector v (i.e., for any vector v such that $Cv = 0$), the work done, at each instant of time, by the scaled constraint force F_s^c must be zero.

The subspace $\mathcal{N}(C)$, at each instant of time t, is simply the tangent plane in which all the scaled virtual displacement vectors reside. Since the scaled force of constraint, at each instant of time, cannot do any work on any scaled virtual displacement vector, there can be no component of the scaled force of constraint on $\mathcal{N}(C)$. But we see by equation (17) that any alteration in the scaled acceleration of the unconstrained motion because of the constraints, is solely engendered by virtue of the scaled force of constraint. Hence the component in $\mathcal{N}(C)$ of the scaled unconstrained acceleration and the scaled constrained acceleration must be identical at each instant of time. This constitutes the second part of our result. Though we have used more general constraints (of the form of equation set (3)) than usually used in analytical mechanics, this result was, in essence, known to LAGRANGE.

The new feature introduced by the recently obtained explicit equation of motion (UDWADIA and KALABA [1]) given by equation (5), is then the knowledge of the explicit magnitude of the component of the scaled constrained acceleration in the subspace $\mathcal{R}(C^T)$, i.e., in the subspace orthogonal to the null space of C. LAGRANGE knew that, at each instant of time, the effect of the constraint was to alter the component of the scaled acceleration in the subspace spanned by the columns of C^T, but he did not provide the explicit magnitude of this component — he signified it as $C^T\lambda$, leaving us his legacy of the ubiquitous Lagrange multiplier λ.

This multiplier m-vector can now be obtained by setting $C^T\lambda = C^+e$, whose solution is simply

$$\lambda = (CC^T)^+ e + [I - (CC^+)^T]\, h, \tag{19}$$

where h is an arbitrary m-vector. Noting that $C = AM^{-1/2}$, we have $CC^T = AM^{-1}A^T$. Using the singular value decomposition of C, the expression for the time dependent Lagrange multiplier m-vector λ now becomes

$$\lambda = (AM^{-1}A^T)^+ e + [I - UU^T]\, h, \tag{20}$$

where the second member on the right hand side obviously lies in the null space of C^T. The Lagrange multiplier λ is, in general, not unique. When the constraints described by the equation set (3) are linearly independent, the rank of the matrix A equals m, and then the second member on the right of equation (20) becomes zero; the Lagrange multiplier m-vector in that case becomes unique. It should be noted that for the general situation, despite the nonuniqueness in the m-vector λ, the equation of motion and the force of constraint remain unique because the second member on the right in equation (19) lies in the null space of C^T.

Since the vector h does not enter the equation of motion describing the constrained system, for the purposes of obtaining this equation we could just as well take h to be zero. The component of the vector λ which lies in the range space of C, indicated by the first member on the right hand side of equation (20), is then what really matters as far as the determination of the equation of motion of the constrained system is concerned. As seen from equation (20), at each instant of time, this component is directly proportional to the vector $\boldsymbol{e}$, the extent to which the acceleration of the unconstrained system does not satisfy the constraints.

It is noteworthy that the matrix of proportionality, $(AM^{-1}A^{\mathrm{T}})^{+}$, is an m by m symmetric matrix. The j-th component λ_j, of the vector $\boldsymbol{\lambda}$, is the scalar Lagrange multiplier corresponding to the j-th constraint equation in the constraint set given by equation (4). Also, the extent to which this j-th constraint equation is dissatisfied because of the presence of the constraints is simply the j-th component, e_j, of the vector $\boldsymbol{e}$. Then the symmetry of $(AM^{-1}A^{\mathrm{T}})^{+}$ implies the following: At each instant of time, the influence on λ_j solely due to the dissatisfaction of the i-th constraint equation in the set (4) by a certain (small) amount, is the same as the influence on λ_i solely due to the dissatisfaction of the j-th constraint equation by that same amount. Thus, at every time t, $\partial\lambda_i \setminus \partial e_j = \partial\lambda_j \setminus \partial e_i$ for $i, j = 1, 2, \ldots, m$.

It is interesting that the geometry of constrained motion — an aspect of Nature that has evaded our grasp for so long — is in fact so simple and elegant. The scaled acceleration vector of the constrained system when projected onto the appropriate subspaces gives rise to components which are, at each instant of time, remarkably simple to determine. One cannot help but marvel at the simplicity and elegance with which Nature seems to operate.

References

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